ON THE ATTACHED PRIME IDEALS OF LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

ZOHREH HABIBI, MARYAM JAHANGIRI, AND KHADIJEH AHMADI AMOLI

Abstract. Let I and J be two ideals of a commutative Noetherian ring R and M be an R-module of dimension d. If R is a complete local ring and M is finite, then attached prime ideals of $H^{d-1}_{I,J}(M)$ are computed by means of the concept of co-localization. Also, we illustrate the attached prime ideals of $H^t_{I,J}(M)$ on a non-local ring R, for $t = \dim M$ and $t = \mathrm{cd}\,(I,J,M)$.

1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring, M an R-module and I and J stand for two ideals of R. For all $i \in \mathbb{N}_0$ the i-th local cohomology functor with respect to (I,J), denoted by $H^i_{I,J}(-)$, defined by Takahashi et. all in [10] as the i-th right derived functor of the (I,J)- torsion functor $\Gamma_{I,J}(-)$, where

$$\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}.$$

This notion coincides with the ordinary local cohomology functor $H^i_I(-)$ when $J = 0$, see [2].

The main motivation for this generalization comes from the study of a dual of ordinary local cohomology modules $H^i_I(M)$ ([9]). Basic facts and more information about local cohomology defined by a pair of ideals can be obtained from [10], [3] and [4].

The second section of this paper is devoted to study the attached prime ideals of local cohomology modules with respect to a pair of ideals by means of co-localization. The concept of co-localization introduced by Richardson in [8].

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Let $(R, \mathfrak{m})$ be local and M be a finite R-module of dimension d. If c is a non-negative integer such that $H^i_{I,J}(R) = 0$ for all $i > c$ and $H^c_{I,J}(R)$ is representable, then we illustrate the attached prime ideals of ${}^{\mathfrak{p}}H^c_{I,J}(M)$ (see Theorem 2.3). In addition if R is complete, then we have made use of Theorem 2.3 to prove that in a special case

$$\mathrm{Att}\,(H^{d-1}_{I,J}(M)) \subseteq T \cup \mathrm{Assh}\,(M) \quad \text{and} \quad T \subseteq \mathrm{Att}\,(H^{d-1}_{I,J}(M)),$$

where

$$T = \{\mathfrak{p} \in \mathrm{Supp}\,(M) : \dim M/\mathfrak{p}M = d-1, J \subseteq \mathfrak{p} \textit{ and } \sqrt{I+\mathfrak{p}} = \mathfrak{m}\},$$

(see Theorem 2.5).

In [3, Theorem 2.1] the set of attached prime ideals of $H^{dimM}_{I,J}(M)$ was computed on a local ring. We generalize this theorem to the non-local case. Also, the authors in [5, 2.4] specified a subset of attached prime ideals of ordinary top local cohomology module $H^{cd(I,M)}_I(M)$. We improve it for $H^{cd(I,J,M)}_{I,J}(M)$ over a not necessarily local ring, where $\mathrm{cd}\,(I, J, M) = \sup\{i \in \mathbb{N}_0 : H^i_{I,J}(M) \neq 0\}$ with the convention that $\mathrm{cd}\,(I, M) = \mathrm{cd}\,(I, 0, M)$.

2. Attached prime ideals

In this section we study the set of attached prime ideals of local cohomology modules with respect to a pair of ideals.

Remark 2.1. Following [8], for a multiplicatively closed subset S of the local ring $(R, \mathfrak{m})$, the co-localization of M relative to S is defined to be the $S^{-1}R$-module $S_{-1}(M) := D_{S^{-1}R}(S^{-1}D_R(M))$, where $D_R(-)$ is the Matlis dual functor $\mathrm{Hom}\,_R(-, E_R(R/\mathfrak{m}))$. If $S = R \setminus \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Spec}\,(R)$, we write ${}^{\mathfrak{p}}M$ for $S_{-1}(M)$.

Richardson in [8, 2.2] proved that if M is a representable R- module, then so is $S_{-1}(M)$ and $\mathrm{Att}\,(S_{-1}M) = \{S^{-1}\mathfrak{p} : \mathfrak{p} \in \mathrm{Att}\,(M)\}$. Therefore, in order to get some results about attached prime ideals of a module, it is convenient to study the attached prime ideals of the co-localization of it.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ be an ideal of R and $\mathfrak{p} \in Spec\,(R)$ with $\mathfrak{a} \subseteq \mathfrak{p}$. Let $R' = R/\mathfrak{a}$ and $\mathfrak{p}' = \mathfrak{p}/\mathfrak{a}$. Then for any R'-module X and $R'_{\mathfrak{p}'}$-module Y, the following isomorphisms hold:*

(i) $D_R(X) \cong D_{R'}(X)$ *as R-modules.*

(ii) $D_R(X)_{\mathfrak{p}} \cong D_{R'}(X)_{\mathfrak{p}'}$ *as $R_{\mathfrak{p}}$-modules.*

(iii) $D_{R_{\mathfrak{p}}}(Y) \cong D_{R'_{\mathfrak{p}'}}(Y)$ *as $R_{\mathfrak{p}}$-modules.*

In [7, 2.1 and 2.2] the following theorems have been proved for the attached prime ideals of $H^d_I(R)$ and $H^{d-1}_I(R)$ where $d = \dim R$. Here, we generalize these theorems for the local cohomology modules of M with respect to a pair of ideals when M is a finite R-module with $\dim M = d$.

Theorem 2.3. *Let $(R, \mathfrak{m})$ be a local ring, M be a finite R-module, and $\mathfrak{p} \in \operatorname{Spec}(R)$. Assume that $c = cd(I, J, R)$ and $H^c_{I,J}(R)$ is representable. Then*

(1) $Att_{R_\mathfrak{p}}({}^\mathfrak{p}H^c_{I,J}(M)) \subseteq \{\mathfrak{q}R_\mathfrak{p} : dim\, M/\mathfrak{q}M \geq c,\ \mathfrak{q} \subseteq \mathfrak{p},\ and\ \mathfrak{q} \in Spec\,(R)\}$.

(2) *If R is complete, then*

$$Att_{R_\mathfrak{p}}({}^\mathfrak{p}H^{dimM}_{I,J}(M)) = \{\ \mathfrak{q}R_\mathfrak{p} : \mathfrak{q} \in Supp\,(M), dim\, M/\mathfrak{q}M = dim\, M, J \subseteq \mathfrak{q} \subseteq \mathfrak{p}, \\ and \sqrt{I + \mathfrak{q}} = \mathfrak{m}\}.$$

Proof. (1) Let $\mathfrak{q}R_\mathfrak{p} \in \operatorname{Att}_{R_\mathfrak{p}}({}^\mathfrak{p}H^c_{I,J}(M))$. By [11, 3.1] and Remark 2.1, we have $H^c_{I,J}(M)$ is representable and $\operatorname{Att}_{R_\mathfrak{p}}({}^\mathfrak{p}H^c_{I,J}(M)) = \{\mathfrak{q}R_\mathfrak{p} : \mathfrak{q} \in \operatorname{Att}(H^c_{I,J}(M))\ and\ \mathfrak{q} \subseteq \mathfrak{p}\}$. Also, using [2, 6.1.8] and [1, 2.11]

$$\operatorname{Att}(H^c_{I,J}(M/\mathfrak{q}M)) = \operatorname{Att}(H^c_{I,J}(M)) \cap \operatorname{Supp}(R/\mathfrak{q}).$$

This implies that $H^c_{I,J}(M/\mathfrak{q}M) \neq 0$ and consequently $\dim M/\mathfrak{q}M \geq c$.

(2) Let $\mathfrak{p} \in \operatorname{Supp}(M)$. Put $d := \dim M$, $\overline{R} = R/\operatorname{Ann}_R M$, and

$$T := \{\mathfrak{q}R_\mathfrak{p} : \mathfrak{q} \in \operatorname{Supp}(M), \dim M/\mathfrak{q}M = d, J \subseteq \mathfrak{q} \subseteq \mathfrak{p}\ and\ \sqrt{I + \mathfrak{q}} = \mathfrak{m}\}.$$

Since $\dim_{\overline{R}} M = \dim_R M$, [10, 2.7] and Lemma 2.2 imply that ${}^{\overline{\mathfrak{p}}}H^d_{I\overline{R},J\overline{R}}(M) \cong {}^\mathfrak{p}H^d_{I,J}(M)$, as $R_\mathfrak{p}$-modules. Therefore, by [2, 8.2.5], $\mathfrak{q} \in \operatorname{Att}_{\overline{R}_{\overline{\mathfrak{p}}}}({}^{\overline{\mathfrak{p}}}H^d_{I\overline{R},J\overline{R}}(M))$ if and only if

$$\mathfrak{q} \cap R_\mathfrak{p} \in \operatorname{Att}_{R_\mathfrak{p}}({}^{\overline{\mathfrak{p}}}H^d_{I\overline{R},J\overline{R}}(M)) = \operatorname{Att}_{R_\mathfrak{p}}({}^\mathfrak{p}H^d_{I,J}(M)).$$

Now, without loss of generality, we may assume that M is faithful and $\dim R = d$. If $H^d_{I,J}(M) = 0$, then $\operatorname{Att}_{R_\mathfrak{p}}({}^\mathfrak{p}H^d_{I,J}(M)) = \emptyset$. Assume that $T \neq \emptyset$ and $\mathfrak{q}R_\mathfrak{p} \in T$. Since $\dim M/\mathfrak{q}M = \dim R$, we have $\dim R/\mathfrak{q} = d$. On the other hand, $\mathfrak{q} \in \operatorname{Supp}(M/JM)$. Thus, by [3, Theorem 2.4], $\dim R/(I + \mathfrak{q}) > 0$ which contradicts $\sqrt{I + \mathfrak{q}} = \mathfrak{m}$. So $T = \emptyset$.

Now, we assume that $H^d_{I,J}(M) \neq 0$.

$\supseteq$: Let $\mathfrak{q}R_\mathfrak{p} \in T$. Since $H^d_{I,J}(M)$ is an Artinian R-module (cf. [4, 2.1]) so, by Remark 2.1, it is enough to show that $\mathfrak{q} \in \operatorname{Att}(H^d_{I,J}(M))$. As $M/\mathfrak{q}M$ is J-torsion with dimension d and $\sqrt{I + \mathfrak{q}} = \mathfrak{m}$, so by [2, 4.2.1 and 6.1.4].

$$H^d_{I,J}(M/\mathfrak{q}M) \cong H^d_I(M/\mathfrak{q}M) \cong H^d_{I(R/\mathfrak{q})}(M/\mathfrak{q}M) \cong H^d_{\mathfrak{m}/\mathfrak{q}}(M/\mathfrak{q}M) \neq 0.$$

Hence [2, 6.1.8] and [1, 2.11] imply that $\emptyset \neq \mathrm{Att}\,(H^d_{I,J}(M/\mathfrak{q}M)) = \mathrm{Att}\,(H^d_{I,J}(M)) \cap \mathrm{Supp}\,(R/\mathfrak{q})$. Let $\mathfrak{q}_0 \in \mathrm{Att}\,(H^d_{I,J}(M))$ be such that $\mathfrak{q} \subset \mathfrak{q}_0$. So that $\dim M/\mathfrak{q}_0 M < d$. On the other hand, by Remark 2.1, $\mathfrak{q}_0 R_{\mathfrak{q}_0} \in \mathrm{Att}\,_{R_{\mathfrak{q}_0}}({}^{\mathfrak{q}_0}H^d_{I,J}(M))$ and this implies that $\dim M/\mathfrak{q}_0 M \geq d$ which is a contradiction. So $\mathfrak{q} = \mathfrak{q}_0$.

$\subseteq$: Let $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Att}\,_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H^d_{I,J}(M))$. As we have seen in the proof of part (1), $\dim M/\mathfrak{q}M = d$ and $\mathfrak{q} \subseteq \mathfrak{p}$. So by [10, 2.7],

$$H^d_{IR/\mathfrak{q},JR/\mathfrak{q}}(M/\mathfrak{q}M) \cong H^d_{I,J}(M/\mathfrak{q}M) \neq 0.$$

Now, by [3, Theorem 2.4], there exists $\mathfrak{r}/\mathfrak{q} \in \mathrm{Supp}\,(R/\mathfrak{q} \otimes_{R/\mathfrak{q}} \frac{M/\mathfrak{q}M}{(JR/\mathfrak{q})(M/\mathfrak{q}M)})$ such that $\dim \frac{R/\mathfrak{q}}{\mathfrak{r}/\mathfrak{q}} = d$ and $\dim \frac{R/\mathfrak{q}}{IR/\mathfrak{q}+\mathfrak{r}/\mathfrak{q}} = 0$. Since $\mathfrak{q}R_{\mathfrak{p}} \in \mathrm{Att}\,_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H^d_{I,J}(M))$, we have $\mathfrak{q} \in \mathrm{Att}\,(H^d_{I,J}(M))$ and so $\mathfrak{q} \in \mathrm{Supp}\,(M) \cap V(J)$. Hence $\mathfrak{q}/\mathfrak{q} \in \mathrm{Supp}\,_{R/\mathfrak{q}}(M/\mathfrak{q}M)$ and then

$$\dim R/\mathfrak{q} = \dim M/\mathfrak{q}M = d = \dim \frac{R/\mathfrak{q}}{\mathfrak{r}/\mathfrak{q}} = \dim R/\mathfrak{q}.$$

Therefore, $\dim R/\mathfrak{q} = \dim R/\mathfrak{r}$ which shows that $\mathfrak{q} = \mathfrak{r}$. Thus $\sqrt{I+\mathfrak{q}} = \mathfrak{m}$. $\square$

Remark 2.4. The inclusion in Theorem 2.3(1) is not an equality in general. Let the assumption be as in Theorem 2.3. Assume that $H^d_{I,J}(M) = 0$, $\mathfrak{p} \in \mathrm{Min}\,(M)$ and $\dim M/\mathfrak{p}M = d$. Then $\mathrm{Att}\,_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H^d_{I,J}(M)) = \emptyset$. But

$$\{\mathfrak{q}R_{\mathfrak{p}} : \dim M/\mathfrak{q}M = d, \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \mathfrak{q} \in \mathrm{Supp}\,(M)\} = \{\mathfrak{p}R_{\mathfrak{p}}\}.$$

Theorem 2.5. *Let $(R,\mathfrak{m})$ be a complete local ring and M be a finite R-module with dimension d. Assume that $H^i_{I,J}(R) = 0$ for all $i > d-1$ and $H^{d-1}_{I,J}(R)$ is representable. Then*

(1)

$$\begin{aligned} \mathit{Att}\,_R(H^{d-1}_{I,J}(M)) \subseteq \quad \{ & \mathfrak{p} \in \mathit{Supp}\,(M) : \mathit{dim}\, M/\mathfrak{p}M = d-1, J \subseteq \mathfrak{p} \text{ and } \sqrt{I+\mathfrak{p}} = \mathfrak{m}\} \\ & \cup \mathit{Assh}\,(M). \end{aligned}$$

(2)

$$\{\mathfrak{p} \in \mathit{Supp}\,(M) : \mathit{dim}\, M/\mathfrak{p}M = d-1, J \subseteq \mathfrak{p} \text{ and } \sqrt{I+\mathfrak{p}} = \mathfrak{m}\} \subseteq \mathit{Att}\,(H^{d-1}_{I,J}(M)).$$

Proof. (1) First we note that, by [10, 4.8] and [11, 3.1], $H^{d-1}_{I,J}(M)$ is representable and $\mathrm{Att}\,(H^{d-1}_{I,J}(M)) \subseteq \mathrm{Supp}\,(M)$. Now, let $\mathfrak{p} \in \mathrm{Att}\,(H^{d-1}_{I,J}(M))$. Since $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Att}\,_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H^{d-1}_{I,J}(M))$, by Theorem 2.3 (1), $\dim M/\mathfrak{p}M \geq d-1$.

If $\dim M/\mathfrak{p}M = d$, then $\dim R/\mathfrak{p} = d$ and so $\mathfrak{p} \in \mathrm{Assh}\,(M)$.

Now, assume that $\dim M/\mathfrak{p}M = d-1$. Since $\mathfrak{p} \in \operatorname{Att}(H^{d-1}_{I,J}(M))$, $H^{d-1}_{IR/\mathfrak{p},JR/\mathfrak{p}}(M/\mathfrak{p}M) \cong H^{d-1}_{I,J}(M/\mathfrak{p}M) \neq 0$. Thus, by [3, Theorem 2.4], there exists $\mathfrak{r}/\mathfrak{p} \in \operatorname{Supp}(\frac{M/\mathfrak{p}M}{(JR/\mathfrak{p})(M/\mathfrak{p}M)})$ such that $\dim \frac{R}{\mathfrak{r}} = d$ and $\dim \frac{R}{I+\mathfrak{r}} = 0$. Hence $\mathfrak{r} = \mathfrak{p}$, $J \subseteq \mathfrak{p}$, and $\sqrt{I+\mathfrak{p}} = \mathfrak{m}$.

(2) Let $\mathfrak{p} \in \operatorname{Supp}(M)$, $J \subseteq \mathfrak{p}$, $\dim M/\mathfrak{p}M = d-1$, and $\sqrt{I+\mathfrak{p}} = \mathfrak{m}$. Then, by [11, 3.1] and Theorem 2.3 (2), $H^{d-1}_{I,J}(M)$ is representable, $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H^{d-1}_{I,J}(M/\mathfrak{p}M))$, and so $\mathfrak{p} \in \operatorname{Att}(H^{d-1}_{I,J}(M/\mathfrak{p}M))$. Now, the proof is complete by considering the epimorphism $H^{d-1}_{I,J}(M) \to H^{d-1}_{I,J}(M/\mathfrak{p}M)$.

□

In the rest of the paper, following [10], we use the notations

$$W(I,J) := \{\mathfrak{p} \in Spec(R) : I^n \subseteq \mathfrak{p}+J \ for\ an\ integer\ n \geq 1\}$$

and

$$\widetilde{W}(I,J) := \{\mathfrak{a} : \mathfrak{a}\ is\ an\ ideal\ of\ R; I^n \subseteq \mathfrak{a}+J \ for\ an\ integer\ n \geq 1\}.$$

The following lemma can be proved using [10, 3.2].

Lemma 2.6. *For any non-negative integer i and R-module M,*

(i) $\operatorname{Supp}(H^i_{I,J}(M)) \subseteq \bigcup_{\mathfrak{a}\in\widetilde{W}(I,J)} \operatorname{Supp}(H^i_{\mathfrak{a}}(M))$.

(ii) $\operatorname{Supp}(H^i_{I,J}(M)) \subseteq \operatorname{Supp}(M) \cap W(I,J)$.

Corollary 2.7. *Let M be an R-module and $c = cd(I,J,R)$. Assume that M is representable or $H^c_{I,J}(R)$ is finite. Then*

$$\operatorname{Att}(H^c_{I,J}(M)) \subseteq \operatorname{Att}(M) \cap W(I,J).$$

Proof. By [10, 4.8], [1, 2.11],[11, 3.1] and Lemma 2.6 (ii), we have

$$\begin{aligned}\operatorname{Att}(H^c_{I,J}(M)) &= \operatorname{Att}(M \otimes H^c_{I,J}(R)) &&\subseteq \operatorname{Att}(M) \cap \operatorname{Supp}(H^c_{I,J}(R)) \\ & &&\subseteq \operatorname{Att}(M) \cap W(I,J).\end{aligned}$$

□

Applying the set of attached prime ideals of top local cohomology module in [3, Theorem 2.2], we obtain another presentation for it.

Proposition 2.8. *Let $(R,\mathfrak{m})$ be a local ring and $\hat{R}$ denotes the $\mathfrak{m}-$adic completion of R. Suppose that M is a finite R-module of dimension d. Then*

$$\begin{aligned}Att_R(H^d_{I,J}(M)) = \ \{ \ & \mathfrak{q} \cap R : \mathfrak{q} \in Supp_{\hat{R}}(\hat{R} \otimes_R M/JM), dim\,(\hat{R}/\mathfrak{q}) = d, \\ & and\ dim\,\hat{R}/(I\hat{R}+\mathfrak{q}) = 0\}.\end{aligned}$$

Proof. Denote the set of right hand side of the assertion by T. It is clear that by[3, Theorem 2.4], $H^d_{I,J}(M) = 0$ if and only if $T = \emptyset$. Assume that $H^d_{I,J}(M) \neq 0$ and $\mathfrak{p} \in \operatorname{Supp}(M/JM)$ with the property that $\operatorname{cd}(I, R/\mathfrak{p}) = d$. Let $\mathfrak{q} \in \operatorname{Ass}(M/JM)$ be such that $\mathfrak{q} \subseteq \mathfrak{p}$. Then

$$d = \operatorname{cd}(I, R/\mathfrak{p}) \leq \operatorname{cd}(I, R/\mathfrak{q}) \leq \dim R/\mathfrak{q} \leq \dim M/JM \leq \dim M = d$$

implies that $\mathfrak{p} = \mathfrak{q} \in \operatorname{Ass}(M/JM)$ and $\dim M/JM = d$. Now the claim follows from [11, 3.10] and [3, Theorem 2.1].

□

The following lemma, which can be proved by using the similar argument of [10, 4.3], will be applied in the rest of the paper.

Lemma 2.9. *Let M be a finite R-module. Suppose that $J \subseteq J(R)$, where $J(R)$ denotes the Jacobson radical of R, and $\dim M/JM = d$ be an integer. Then $H^i_{I,J}(M) = 0$ for all $i > d$.*

Using Lemma 2.9, we can compute $\operatorname{Att}(H^{dimM}_{I,J}(M))$ in non-local case as a generalization of [6, 2.5].

Proposition 2.10. *Let M be a finite R-module of dimension d and $J \subseteq J(R)$. Then*

$$\begin{aligned} \operatorname{Att}(H^d_{I,J}(M)) &= \operatorname{Att}(H^d_I(M/JM)) \\ &= \{\mathfrak{p} \in \operatorname{Ass}(M) \cap V(J) : \operatorname{cd}(I, R/\mathfrak{p}) = d\}. \end{aligned}$$

Proof. The assertion holds by applying Lemma 2.9 and using the same method of the proof of [3, Theorem 2.1 and Proposition 2.1]. □

Corollary 2.11. *Suppose that $J \subseteq J(R)$ and M is a finite R-module such that $\dim M = d$. Then*

$$\operatorname{Att}\left(\frac{H^d_{I,J}(M)}{JH^d_{I,J}(M)}\right) = \{\mathfrak{p} \in \operatorname{Supp}(M) \cap V(J) : \operatorname{cd}(I, R/\mathfrak{p}) = d\}.$$

Proof. Let $\overline{R} = R/\operatorname{Ann}_R M$. Using [10, 2.7], $H^d_{I,J}(M) \cong H^d_{I\overline{R},J\overline{R}}(M)$ and also for a prime $\mathfrak{p} \in \operatorname{Supp}(M) \cap V(J)$, $\operatorname{cd}(I\overline{R}, \overline{R}/\mathfrak{p}) = \operatorname{cd}(I, R/\mathfrak{p})$. Thus we may assume that M is faithful and so $\dim R = d$. In virtue of [2, 6.1.8], $H^d_I(M/JM) \cong H^d_{I,J}(M/JM) \cong \frac{H^d_{I,J}(M)}{JH^d_{I,J}(M)}$. Now, the assertion follows by Proposition 2.10. □

The final result of this section is a generalization of [5, 2.4] in non-local case for local cohomology modules with respect to a pair of ideals.

Proposition 2.12. *Let $J \subseteq J(R)$ and M be a finite R-module. Then*

$$\{\mathfrak{p} \in Ass(M) \cap V(J) : cd(I, R/\mathfrak{p}) = dim\, R/\mathfrak{p} = cd(I, J, M)\} \subseteq Att(H_{I,J}^{cd(I,J,M)}(M)).$$

Equality holds if $cd(I, J, M) = dim\, M$.

Proof. The same proof of [5, 2.4] remains valid by using Proposition 2.10. □

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Zohreh Habibi, Payame Noor University, Po Box 19395-3697, Tehran, Iran
E-mail address: z_habibi@pnu.ac.ir

Maryam Jahangiri, Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran AND Institute for Research in Fundamental Sciences (IPM) P. O. Box: 19395-5746, Tehran, Iran.
E-mail address: jahangiri@khu.ac.ir

Khadijeh Ahmadi Amoli, Payame Noor University, Po Box 19395-3697, Tehran, Iran
E-mail address: khahmadi@pnu.ac.ir